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G.R.I.M. (Departimento di Matematica e Informatica, University of Palermo, Italy)

Is this a proof? Future teachers' conceptions of proof

Alexandra Gomes, CIEC/IE, University of Minho, magomes@ie.uminho.pt
Floriano Viseu, CIEd/IE, University of Minho, fviseu@ie.uminho.pt
Paula Mendes Martins, CMAT, University of Minho, pmendes@math.uminho.pt
Luís Menezes, CI&DETS, Polytechnic of Viseu, menezes@esev.ipv.pt

Abstract: In this paper we present part of an ongoing investigation that aims at disclosing the conceptions of proof held by future elementary school teachers. Using a qualitative and interpretative approach, we analyzed data from 66 questionnaires and results show that almost all participants recognize the formal aspect of an algebraic proof but they also accept some examples as proof.

Résumé: Dans cet article, nous présentons partie d'une enquête en cours qui vise à révéler les conceptions de la preuve détenue par les futurs enseignants du primaire. En utilisant une approche qualitative et interprétative, nous avons analysé les données de 66 questionnaires et les résultats montrent que presque tous les participants reconnaissent l'aspect formel d'une preuve algébrique mais ils acceptent aussi quelques exemples comme preuve.

Introduction

Proof plays a fundamental role in the construction of mathematical knowledge, having a different nature and assuming a different structure that in other sciences (Davis & Hersh, 1995; Dreyfus, 2000; Hanna, 2000; Knuth, 2002). To prove is intrinsic to mathematical activity because no result can be considered valid/acceptable until it is proven. Assuming a formal, logical view, proof can be defined as "a sequence of transformations of formal sentences, carried out according to the rules of the predicate calculus" (Hersh, 1993, p. 391). But proof can also be regarded in a more practical manner as "an argument that convinces qualified judges" (ibidem). Therefore, for the same result we can have different proofs. Some proofs have a geometric approach, some are more algebraic, and some have only words while others have only diagrams. Thus, what characteristics must a proof have to be considered as such?

Also important is to consider why proof is used. There are several functions attributed to proof (De Villiers, 2003). In teaching and learning mathematics two functions tend to be predominant: conviction and explanation (Hanna, 2000; Hersh, 1993). The importance of proof in the classroom, especially in the early years, has not always been recognized (Hanna, 2000). Proof appears more linked to secondary and higher education and its' understanding is sometimes referred to the good students only (Knuth, 2002). However, several authors have highlighted the role of proof in the construction of mathematical knowledge by students from the beginning of schooling (Bussi, 2009; Hanna, 2000; Knuth, 2002; Stylianides, 2007). Proof of mathematical results comes prized in the current reformulation of programs from the early grades, in Portugal (ME, 2007; MEC, 2013). From the early grades, children should start to learn and to deal with proof and proving. For this to happen it is important that teachers develop strategies to motivate and encourage students for the activity of proving showing them the power/importance of proof and do not reduce proof to a mere memorization of sequential meaningless steps. This process is clearly dependent on the perception that teachers have of proof and of what it means to prove. Therefore, teachers need to be prepared to deal with proof and to have a clear understanding about proof and proving. Consequently, during initial teacher training, future teachers should be involved in proof activities that enable them to deepen their knowledge on proof, and enhance their ability to validate, organize, justify and generalize acquired mathematical knowledge.

Methodology

The purpose of this study was to disclose the conceptions held by future elementary school teachers (grades 1 through 6) concerning the notion of proof. Nowadays, in Portugal, in order to become an elementary school teacher, one has to have a 3 years degree in Basic Education and then take a Masters degree in teaching. These teachers have to teach several subjects; therefore, the curriculum of these degrees is very wide and covers a great variety of topics.

The participants on this study were 66 students enrolled in the 3rd year of a Basic Education Degree. These participants had already taken 5 courses in mathematics during which they had some contact with proof, performing proofs of some mathematical results, such as the irrationality of the square root of two, Pythagoras theorem or the constant sum of the distances from a point inside a triangle.

Given the nature of the outlined goal, we adopted a qualitative and interpretative approach in order to understand the meaning that future teachers give to this activity (Bogdan & Biklen, 1994).

We designed a short questionnaire containing four mathematical results and, for each, we gave three proposals of proof. We then asked the students to say whether they considered each proposal to be a proof or not and to justify their choices. This questionnaire was completed individually, at the end of a regular class, taught by one of the authors. The participation was voluntary and could be anonymous, that is, students only identified themselves if they wanted to.

For this paper we selected the following two results: (1) Diagonals of a rhombus are perpendicular; (2) For $a, b \in R, (a+b)^2 = a^2 + 2ab + b^2$.

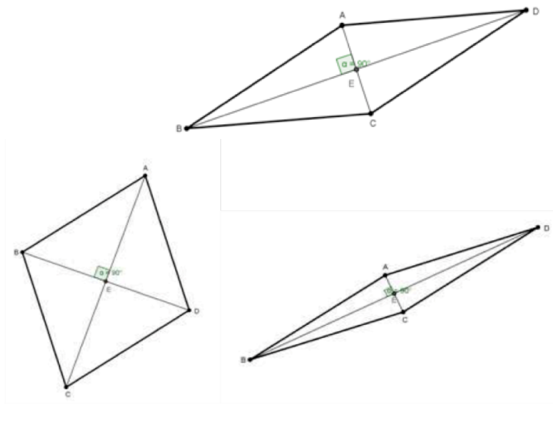
Some findings

Result 1

Diagonals of a rhombus are perpendicular. For this result, students were confronted with the 3 proposals of proof.

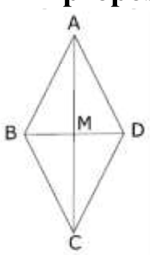
1st proposal of proof

Draw the diagonals [AC] and [BD] and consider the intersection point (E). Measuring one of the angles defined by the diagonals we easily conclude that the diagonals are perpendicular. Now we check that the same happens for other rhombus, in other positions and other shapes:



In these cases, measuring the angles, we also check that the diagonals are perpendicular. For many other different rhombus we would come to the same conclusion. Hence, we have proved that the diagonals of a rhombus are perpendicular.

2nd proposal of proof



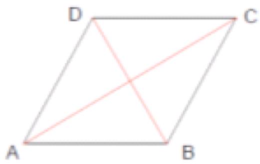
Let [ABCD] be an rhombus represented as follows:

[AC] and [BD] are the diagonals of [ABCD]. M is the midpoint of [AC] and of [BD].

[ABD] is an isosceles triangle, then [AM] is the height of the triangle relatively to the side [BD]. [AC] is perpendicular to [BD].

3rd proposal of proof

Let [ABCD] be an rhombus represented as follows:



Since $\overline{BC} = \overline{DC}$, we have that the point C belongs to the perpendicular bisector of [BD]. Since $\overline{AB} = \overline{AD}$, we have that the A belongs to the perpendicular bisector of [BD].

Hence, the line segment [AC] is contained in the perpendicular bisector of the line segment of [BD]. Therefore, [BD] and [AC] are perpendicular.

The choices made by the students were the following:

	Is proof	Is not proof	No answer	Total
Proposal 1: Examples	45	18	3	66
Proposal 2: Loci	23	40	3	66
Proposal 3: Measurement	32	31	3	66

Table 1. Frequency of student answers to each of the proposals.

For most students, the verification, in individual cases, of geometric results to prove is seen as a proof. The same students have difficulty following an argument that works only with loci and so, although it is a proof, the second proposal was not considered as such. Working with measures, there is a balance between the number of students that identifies the proposal as evidence and as no proof.

Analysing the reasons given for proposal 1, we found the following results:

	Is proof		**Is not proof**	
	Particular cases allow generalization	28	Particular cases don't allow generalization	10
	Use the result	15	No justification	2
	Meaningless justification	2	Meaningless justification	6
Total		45		18

Table 2. Justifications provided in response to proposal 1.

Examining the reasons given for proposal 1, we found that the majority of students said that proposal 1 was a proof and justified by saying that particular cases could be generalized. We illustrate this with an example of a justification given by one of the students:

This case is a mathematical proof, because we started with a case and found a conjecture. After we used further examples to see if it works in order to generalize.

Interestingly, the justification used for saying that it was not a proof is opposite: particular cases don't allow generalization, as another student referred:

It is not a mathematical proof, since it makes use of particular cases.

A significant number of students that said it was a proof, also justified using the result, as shown in this students' answer:

In my opinion, this proposal 1 is according to the result since drawing two diagonals, and finding the intersection point on the rhombus, they are always perpendicular.

Looking at the reasons given for proposal 2, we found the following results:

<table>
<tr><td rowspan="6"></td><td colspan="2">Is proof</td><td colspan="2">Is not proof</td></tr>
<tr><td>Use the result</td><td>6</td><td>Diagram as particular case</td><td>28</td></tr>
<tr><td>Identifies a correct reasoning</td><td>5</td><td>Insufficient data</td><td>5</td></tr>
<tr><td>Meaningless justification</td><td>6</td><td>Meaningless justification</td><td>5</td></tr>
<tr><td>No justification</td><td>2</td><td rowspan="2">No justification</td><td rowspan="2">2</td></tr>
<tr><td>Validation of a step of reasoning</td><td>4</td></tr>
<tr><td>Total</td><td colspan="2">23</td><td colspan="2">40</td></tr>
</table>

Table 3. Justifications provided in response to proposal 2.

The majority of the students said that it was not a proof and among these, more than half justified their opinion saying that they considered the drawing/diagram to be a particular case. An example of the justifications given is:

If [AC] *is perpendicular to* [BD] *then this rhombus has its diagonals perpendicular. However this is only a particular case, it does not occur for all rhombi.*

Some of those saying it was a proof, used the result itself as justification:

Proves because the diagonal pass through the point M and have the same angle.

Others said it was a correct reasoning and others justified the all proof identifying just a correct step, as the following example:

This is a proof since in an isosceles triangle the height of the triangle is perpendicular to its base.

As for proposal 3, the opinions are divided.

<table>
<tr><td rowspan="6"></td><td colspan="2">Is proof</td><td colspan="2">Is not proof</td></tr>
<tr><td>Identify a correct reasoning</td><td>8</td><td>Diagram as particular case</td><td>20</td></tr>
<tr><td>Validation of a step of reasoning</td><td>7</td><td>Meaningless justification</td><td>6</td></tr>
<tr><td>Use the result</td><td>3</td><td>Insufficient data</td><td>3</td></tr>
<tr><td>Meaningless justification</td><td>9</td><td rowspan="2">No justification</td><td rowspan="2">2</td></tr>
<tr><td>No justification</td><td>5</td></tr>
<tr><td>Total</td><td colspan="2">32</td><td colspan="2">31</td></tr>
</table>

Table 4. Justifications provided in response to proposal 3.

As before, many of those who consider that it is not a proof assume the diagram as a particular case. One of the students wrote:

Does not prove, because we have not proven that we can generalize what is happening with this case.

Also, some of the students that said it was a proof, used the result as justification:

It is a proof, because [AC] *and* [BD] *intersected themselves and form a right angle*.

There are students that considered the proposal to be a proof since identified a correct reasoning:

This is a proof since it is obtained by a sequence of the mathematical statements that no one can refute.

Result 2

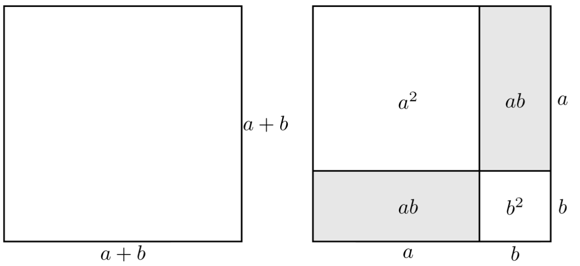
Considering the result: "$a, b \in R, (a+b)^2 = a^2 + 2ab + b^2$", we provided the following proof proposals:

1st proposal of proof

Consider, for example, $a = 2$ and $b = 1$. Then, $(a+b)^2 = 3^2 = 9$ and
$a^2 + 2ab + b^2 = 2^2 + 2 \times 2 \times 1 + 1^2 = 4 + 4 + 1 = 9$. If we consider $a = -1$ and $b = \frac{1}{2}$, we have $(a+b)^2 = (-\frac{1}{2})^2 = \frac{1}{4}$ and
$a^2 + 2ab + b^2 = (-1)^2 + 2 \times (-1) \times \frac{1}{2} + (\frac{1}{2})^2 = 1 - 1 + \frac{1}{4} = \frac{1}{4}$.

For any real values considered for a and b, we will obtain equal values for both members of the equality. Hence, the equality is true.

2nd proposal of proof



3rd proposal of proof

Let a and b be real numbers. Then

$$
\begin{aligned}
(a+b)^2 &= (a+b)(a+b) && \text{[Definition of powers]} \\
&= a(a+b) + b(a+b) && \text{[Distributivity of } \times \text{ over } + \text{]} \\
&= aa + ab + ba + bb && \text{[Distributivity of } \times \text{ over } + \text{]} \\
&= a^2 + ab + ab + b^2 && \text{[Definition of powers and commutativity of } \times \text{]} \\
&= a^2 + 2ab + b^2 .
\end{aligned}
$$

Students chose whether each proposal was a proof or not, in the subsequent manner:

	Is proof	Is not proof	No answer
Proposal 1: Examples	35	30	1
Proposal 2: Without words	26	38	2
Proposal 3: Algebraic	60	4	2

Table 5. Frequency of student answers to each of the proposals(n=66).

In proposal 1, slightly more than half the students said it was a proof.

	Is proof		**Is not proof**	
	Formula is checked in particular cases	32	Particular cases don't allow generalization	24
	Use the result	2	No justification	1
	Meaningless justification	1	Meaningless justification	5
Total		35		30

Table 6. Justifications provided in response to proposal 1.

Almost all justifications were the same as in proposal 1 from the 1st result, that is, particular cases allow verifying the truth of the statement. One model of this was:
Yes, through these examples we can consider that the equality is always true.

For proposal 2 we have the following results:

	Is proof		**Is not proof**	
	Identify the figures to complete equality	14	Insufficient data	28
	Check with concrete values	2		
	Use the result	2		
	Meaningless justification	6	Diagram as particular case	8
	No justification	2	No justification	2
Total		26		38

Table 7. Justifications provided in response to proposal 2.

In this proposal, more students said it wasn't a proof since most of them considered that data was insufficient. An example of a justification given by students is:
It's not a proof, as is not accompanied with any explanation / justification.

Nevertheless, half of the students that considered this proposal to be a proof did it because they could read the figures to complete the equality:

It proves because both figures represent the same number.

Finally, almost all students consider proposal 3 a proof.

	Is proof		**Is not proof**	
	Identify a correct reasoning	29	Insufficient data	3
	Validation of a step of reasoning	8		
	Taken as a generalizable example	9		
	Meaningless justification	5	Meaningless justification	1
	No justification	4		
	Use the result	4		
	Verify each step with values	1		
Total		60		4

Table 8. Justifications provided in response to proposal 3.

The justification given by most of them is that they recognize a well-justified reasoning. For example:
It is a mathematical proof as it demonstrates the equality through mathematical properties and definitions for any real numbers a and b.

There are some students that see this proposal as a generalizable example. One such answer was:

Yes, through this mathematical proof it is possible to generalize.

As before, there are some students that only gave importance to one of the steps of the proof:

It is a proof because it uses the distributive law of the multiplication to adition for real numbers.

Conclusions and Implications

The majority of the students accept that a mathematical result may be proven using a few examples. They tend to evaluate specific cases to ascertain the truth of a result. This finding is corroborated by several researches (Harel & Sowder, 1998). Even though there is some sense that to prove, one needs to generalize, students fail to give a valid argumentation that supports the generalization or they fail to recognize that only giving some examples doesn't necessarily mean that the result is valid. They still think empirically/inductively. This result points to the need of providing opportunities for these students to evolve from this inductive stage.

Some students look at diagrams, in geometry, as particular cases and not as generic as they are intended to be. For them, diagrams seem to represent particular, concrete objects, so they haven't acquired the figural concept (Fischbein, 1993). On the other hand, this result may also be due to the lack of activities during the school trajectory of students that consider different representations of mathematical proof in the results. (Hanna, 2000)

Several students fail to see the need for proof since they simply use the result as a justification. This apparent lack of curiosity for why such result is true seems to indicate that students consider that proof is something irrelevant, meaningless that they just need to memorize.

Almost all students recognize the formal aspect of an algebraic proof. They seem to pay more attention to the formal aspect of a proof than to its' correctness.

These results point to the necessity of rethinking the way proof is approached in the initial teacher training courses. The training of these future teachers should emphasize the understanding and the process of proving rather than the memorization and replication of proofs. Future teachers should appreciate the role of proof and be challenged through appropriate tasks to evolve from the inductive stage. They also should be given opportunities to select and use various types of reasoning and methods of proof.

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